

RESOLUTIONS OVER KOSZUL ALGEBRAS

EDWARD L. GREEN, GREGORY HARTMAN, EDUARDO N. MARCOS, AND ØYVIND SOLBERG

ABSTRACT. In this paper we show that if $\Lambda = \prod_{i \geq 0} \Lambda_i$ is a Koszul algebra with Λ_0 isomorphic to a product of copies of a field, then the minimal projective resolution of Λ_0 as a right Λ -module provides all the information necessary to construct both a minimal projective resolution of Λ_0 as a left Λ -module and a minimal projective resolution of Λ as a right module over the enveloping algebra of Λ . The main tool for this is showing that there is a comultiplicative structure on a minimal projective resolution of Λ_0 as a right Λ -module.

INTRODUCTION AND PRELIMINARIES

Let $\Lambda = \prod_{i \geq 0} \Lambda_i$ be a Koszul algebra over a field k with Λ_0 a product of copies of k , where we recall the definition of Koszul later in this section. Denote by (\mathbb{L}, e) a minimal (graded) projective resolution of Λ_0 as a right Λ -module. We show that (\mathbb{L}, e) contains all the information needed to construct a minimal projective resolution of Λ as a right Λ^e -module, where $\Lambda^e = \Lambda^{\text{op}} \otimes_k \Lambda$. The resolution (\mathbb{L}, e) is shown to have a “comultiplicative structure”. This structure is used to prove that one can obtain a minimal projective resolution of Λ_0 over Λ as a left Λ -module from the knowledge of (\mathbb{L}, e) . We apply these results to prove an unpublished result of E. L. Green and D. Zacharia that Λ is a Koszul algebra if and only if Λ is a linear module as a right module over Λ^e . In [2], the comultiplicative structure is applied to give the multiplicative structure of the Hochschild cohomology ring of a Koszul algebra and also the structure constants for a basis for the Koszul dual.

The rest of the section is devoted to recalling definitions, results, and terminology relevant to this paper. Let $\Lambda = \prod_{i \geq 0} \Lambda_i$ be a graded algebra over a field k . Assume that (i) Λ_0 is a product of copies of k , that (ii) each Λ_i is finite dimensional over k , and that (iii) Λ as an algebra is generated in degrees 0 and 1. Such an algebra Λ is isomorphic to a quotient of the path algebra kQ/I , where kQ is isomorphic to the tensor algebra $T_{\Lambda_0}(\Lambda_1) = \prod_{i \geq 0} \underbrace{\Lambda_1 \otimes_{\Lambda_0} \cdots \otimes_{\Lambda_0} \Lambda_1}_i$. Conversely, if Q is a quiver and

I is an ideal generated by length homogeneous elements in kQ , then $\Lambda = kQ/I$ is a graded algebra over k satisfying the conditions above. Throughout this paper Λ denotes a graded algebra having properties (i)–(iii).

Let $\mathfrak{r} = \prod_{i \geq 1} \Lambda_i$, which is the graded Jacobson radical of Λ . If (\mathbb{P}, d) :

$$\cdots \rightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \rightarrow 0$$

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is a graded projective resolution of a graded Λ -module M , then it is *minimal* if $\text{Im } d^n \subseteq \mathfrak{r}P^{n-1}$ for $n \geq 1$. It is well known that graded modules over graded algebras have minimal graded projective resolutions. We say that a graded projective resolution

$$\cdots \rightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \rightarrow 0$$

is *linear*, and M is a *linear module* if, for $n \geq 0$, the graded module P^n is generated in degree n . Note that a linear resolution is a minimal projective resolution. A graded algebra Λ is a *Koszul algebra* if Λ_0 is a linear module; that is, Λ_0 has a linear (graded) projective resolution (\mathbb{L}, e) :

$$\cdots \rightarrow L^2 \xrightarrow{e^2} L^1 \xrightarrow{e^1} L^0 \xrightarrow{e^0} \Lambda_0 \rightarrow 0$$

as a right Λ -module.

Before giving the precise results, we introduce notation and recall results from [5] which are used throughout the paper. For ease of notation, let $R = kQ$, let \mathcal{B} be the set of all paths in the quiver Q , and denote by \mathcal{B}_t all the paths of length t .

There exist integers $\{t_n\}_{n \geq 0}$ and elements $\{f_i^n\}_{i=0}^{t_n}$ in R such that a minimal right projective resolution (\mathbb{L}, e) of Λ_0 can be given in terms of a filtration of right ideals

$$\cdots \subseteq \Pi_{i=0}^{t_n} f_i^n R \subseteq \Pi_{i=0}^{t_{n-1}} f_i^{n-1} R \subseteq \cdots \subseteq \Pi_{i=0}^{t_1} f_i^1 R \subseteq \Pi_{i=0}^{t_0} f_i^0 R = R$$

in R . Then $L^n = \Pi_{i=0}^{t_n} f_i^n R / \Pi_{i=0}^{t_{n-1}} f_i^{n-1} R$ and the differential e is induced by the inclusion $\Pi_{i=0}^{t_n} f_i^n R \subseteq \Pi_{i=0}^{t_{n-1}} f_i^{n-1} R$. This inclusion gives elements $h_{ji}^{n-1,n}$ in R such that

$$f_i^n = \sum_{j=0}^{t_{n-1}} f_j^{n-1} h_{ji}^{n-1,n}$$

for all $i = 0, 1, \dots, t_n$ and all $n \geq 1$, so that

$$e^n(\overline{f_i^n}) = (\overline{h_{0i}^{n-1,n}}, \overline{h_{1i}^{n-1,n}}, \dots, \overline{h_{t_{n-1}i}^{n-1,n}})$$

for all $n \geq 1$, where $\overline{*}$ denotes the natural residue class of $*$ modulo I . It is shown in [5] that the f_i^n 's can be chosen so that (\mathbb{L}, e) is a minimal resolution of Λ_0 over Λ . We point out that an algorithmic construction of the elements f_i^n 's can be found in [4].

An important property of the elements $\{f_i^n\}_{i=0}^{t_n}$ is that there exist elements $f_j^{n+1'}$ in $\Pi_{i=0}^{t_{n-1}} f_i^n I$ such that

$$(\Pi_{i=0}^{t_n} f_i^n R) \cap (\Pi_{i=0}^{t_{n-1}} f_i^n I) = (\Pi_{i=0}^{t_{n+1}} f_i^{n+1} R) \amalg (\Pi_j f_j^{n+1'} R).$$

Recall that an element x in R is called *uniform* if x is non-zero and there exist vertices u and v in Q such that $x = uxv$. If x is a uniform element with $x = uxv$, then we write $\mathfrak{o}(x) = u$ and $\mathfrak{t}(x) = v$. The elements f_i^n can all be chosen uniform for $i = 0, 1, \dots, t_n$ and all $n \geq 0$, and we assume that they are.

Note that $t_0 + 1$ is the number of non-isomorphic graded simple right Λ -modules, and that $\{f_i^0\}_{i=0}^{t_0}$ is the set of vertices of Q . Moreover, $t_1 + 1$ is the number of arrows of Q and $\{f_i^1\}_{i=0}^{t_1}$ is chosen to be the set of arrows of Q . The set $\{f_i^2\}_{i=0}^{t_2}$ is a set of uniform length homogeneous minimal generators for I .

In case Λ is a Koszul algebra, we have the following additional property of the elements f_i^n in R ; namely each f_i^n is a linear combination of paths in \mathcal{B}_n for

$i = 0, 1, \dots, t_n$ and the length of each path occurring in $f_i^{n'}$ is at least $n + 1$. By length considerations, $h_{ji}^{n-1, n}$ are all linear combinations of elements in \mathcal{B}_1 .

In section 1 we prove that the elements $\{f_i^n\}_{i=0, n \geq 0}^{t_n}$ have the following “comultiplicative structure”, which is used in [2] to give the multiplicative structure of the Hochschild cohomology ring of a Koszul algebra and the structure constants for the basis associated to the elements $\{f_i^n\}$ for the Koszul dual.

Theorem. *Let $\Lambda = kQ/I$ be a Koszul algebra. Then for each r , with $0 \leq r \leq n$, and i , with $0 \leq i \leq t_n$, there exist elements $c_{pq}(n, i, r)$ in k such that*

$$f_i^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n, i, r) f_p^r f_q^{n-r}$$

for all $n \geq 1$, all i in $\{0, 1, \dots, t_n\}$ and all r in $\{0, 1, \dots, n\}$.

Viewing Λ_0 as a left module over Λ , it also has a minimal graded projective resolution given by $\{g_i^n\}_{i=0}^{s_n}$, where g_i^n 's are the left analogue of the right f_j^n 's in R . The above result is used to prove that one can choose the elements g_i^n 's to be the same as the elements f_j^n 's and then the formula $f_i^n = \sum_{p,q=0}^{t_1, t_{n-1}} c_{pq}(n, i, 1) f_p^1 f_q^{n-1}$ gives the differential in the projective resolution of Λ_0 as a left Λ -module. Thus the knowledge of the minimal projective resolution (\mathbb{L}, e) via the elements $\{f_i^n\}$ contains all the information needed to construct a minimal projective resolution of Λ_0 as a left Λ -module.

In the final section of the paper, the elements f_i^n 's are shown to provide all the information needed to construct a minimal projective resolution of Λ as a right Λ^e -module. In particular, we prove the following.

Theorem. *Let $\Lambda = kQ/I$ be a Koszul algebra, and let $\{f_i^n\}_{i=0}^{t_n}$ be defined as above for Λ_0 as a right Λ -module. A minimal projective resolution (\mathbb{P}, δ) of Λ over Λ^e is given by*

$$P^n = \Pi_{i=0}^{t_n} \Lambda \mathfrak{o}(f_i^n) \otimes_k \mathfrak{t}(f_i^n) \Lambda$$

for $n \geq 0$, where j -th component of the differential $\delta^n: P^n \rightarrow P^{n-1}$ applied to the i -th generator $\mathfrak{o}(f_i^n) \otimes \mathfrak{t}(f_i^n)$ is given by

$$\sum_{p=0}^{t_1} c_{pj}(n, i, 1) \overline{f_p^1} \mathfrak{o}(f_j^{n-1}) \otimes \mathfrak{t}(f_j^{n-1}) + (-1)^n \sum_{q=0}^{t_1} c_{jq}(n, i, n-1) \mathfrak{o}(f_j^{n-1}) \otimes \mathfrak{t}(f_j^{n-1}) \overline{f_q^1}$$

for $j = 0, 1, \dots, t_{n-1}$ and $n \geq 1$, and $\delta^0: \Pi_{i=0}^{t_0} \Lambda e_i \otimes_k e_i \Lambda \rightarrow \Lambda$ is the multiplication map.

As mentioned earlier, the final result of the paper is that Λ is a Koszul algebra if and only if Λ is a linear module as a right Λ^e -module.

1. A RESOLUTION WITH COMULTIPLICATIVE STRUCTURE

In this section Theorem 1.1 provides a comultiplicative structure to a minimal projective resolution of Λ_0 as a right Λ -module. This result is then applied to show that the knowledge of a minimal projective resolution of Λ_0 as a right Λ -module is sufficient to construct a minimal projective resolution of Λ_0 as a left Λ -module.

Let $\Lambda = kQ/I$ be a graded algebra over a field k . Let $\{t_n\}_{n \geq 0}$ and $\{f_i^n\}_{i=0, n \geq 0}^{t_n}$ be as in the introduction. We say that $\{f_i^n\}_{i=0, n \geq 0}^{t_n}$ defines a minimal resolution if the resolution described in the introduction is minimal.

The next result shows that the elements $\{f_i^n\}$ have a comultiplicative structure for a Koszul algebra.

Theorem 1.1. *Let $\Lambda = kQ/I$ be a Koszul algebra, and assume that $\{f_i^n\}_{i=0}^{t_n}$ defines a minimal resolution of Λ_0 as a right Λ -module. Then for each r , with $0 \leq r \leq n$, and i , with $0 \leq i \leq t_n$, there exist elements $c_{pq}(n, i, r)$ in k such that*

$$f_i^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n, i, r) f_p^r f_q^{n-r}.$$

Proof. For any n , and r equal to 0 or n , the result follows from $f_i^n = f_i^n f_{\iota(f_i^n)}^0 = f_{\sigma(f_i^n)}^0 f_i^n$ for $i = 0, 1, \dots, t_n$. Also, this proves the result in the case n is equal to 1.

Next we discuss the case $n = 2$. As we have remarked, each $f_i^2 = \sum_{j=0}^{t_1} f_j^1 h_{ji}^{1,2}$. Since Λ is Koszul, each f_i^2 is a linear combination of paths in \mathcal{B}_2 , and hence $h_{ji}^{1,2}$ is a linear combination of elements in \mathcal{B}_1 . This gives the result for $n = 2$.

Now we proceed by induction on n and assume that the result is true for $l < n$ and $n \geq 3$. We have that $f_i^n = \sum_{j=0}^{t_{n-1}} f_j^{n-1} h_{ji}^{n-1,n}$. As in our discussion for $n = 2$, we see that $h_{ji}^{n-1,n}$ is a linear combination of elements in \mathcal{B}_1 . There exist elements c_{ijs} in k such that

$$f_i^n = \sum_{j=0}^{t_{n-1}} \sum_{s=0}^{t_1} c_{ijs} f_j^{n-1} f_s^1.$$

By induction, there exist elements c'_{juv} in k such that

$$f_j^{n-1} = \sum_{u=0}^{t_r} \sum_{v=0}^{t_{n-r-1}} c'_{juv} f_u^r f_v^{n-r-1}$$

for any r , with $0 \leq r \leq n-1$. Hence

$$f_i^n = \sum_j \sum_s \sum_u \sum_v c_{ijs} c'_{juv} f_u^r f_v^{n-r-1} f_s^1$$

for any r , with $0 \leq r \leq n-1$. The term after f_u^r is

$$(1) \quad A = \sum_j \sum_s \sum_v c_{ijs} c'_{juv} f_v^{n-r-1} f_s^1.$$

Theory tells us that

$$f_i^n = \sum_{w=0}^{t_{n-2}} f_w^{n-2} z_w,$$

where z_w is in I . Again by length considerations each z_w is a linear combination of f_l^2 's. Hence, there exist elements c''_{iwx} in k such that

$$f_i^n = \sum_{w=0}^{t_{n-2}} \sum_{x=0}^{t_2} c''_{iwx} f_w^{n-2} f_x^2.$$

By induction each f_w^{n-2} is a linear combination of $f_u^r f_y^{n-r-2}$. We obtain

$$f_i^n = \sum_w \sum_x \sum_u \sum_y c''_{iwx} c'''_{wuy} f_u^r f_y^{n-r-2} f_x^2$$

for some c'''_{wuy} in k . So the term after f_u^r in this expression is

$$(2) \quad B = \sum_w \sum_x \sum_u c''_{iwx} c'''_{wuy} f_y^{n-r-2} f_x^2.$$

Since $\sum_u f_u^r R$ is a direct sum, we see that formulas (1) and (2) are equal. The equation (1) implies that A is in $\Pi_{v=0}^{t_{n-r-1}} f_v^{n-r-1} R$, and the equation (2) implies that A is in $\Pi_{y=0}^{t_{n-r-2}} f_y^{n-r-2} I$. It follows that A is contained in $(\Pi_{t=0}^{t_{n-r}} f_t^{n-r} R) \amalg (\Pi_l f_l^{n-r'} R)$. By length arguments we infer that A is in $\Pi_{t=0}^{t_{n-r}} f_t^{n-r} R$ and that A is a k -linear combination of the f_t^{n-r} 's. Hence we conclude that f_i^n is a k -linear combination of $f_s^r f_t^{n-r}$, and this completes the proof of the result. \square

Since the maps in the minimal projective resolution of Λ_0 as a right Λ -module are given by the $h_{ji}^{n-1,n}$, we explicitly point out the following relationship.

Corollary 1.2. *Keeping the notation of Theorem 1.1, we have*

$$h_{ji}^{n-1,n} = \sum_{l=0}^{t_1} f_l^1 c_{jl}(n, i, n-1)$$

for $n \geq 1$, and i and j , with $0 \leq i \leq t_n$, and $0 \leq j \leq t_{n-1}$.

Before applying Theorem 1.1 we need the following lemma, where J denotes the ideal generated by the arrows in Q .

Lemma 1.3. *Let $\{x_i\}_{i \in \mathcal{I}}$ be a set of elements in the linear span of \mathcal{B}_s . Suppose that $\{x_i\}_{i \in \mathcal{I}}$ is linearly independent viewed as vectors over k . Then $\sum_{i \in \mathcal{I}} R x_i$ and $\sum_{i \in \mathcal{I}} x_i R$ are direct sums.*

Proof. Suppose that $\sum_i \sum_{j \in \mathcal{I}} c_{ij} q_{ij} x_j = 0$ in R for some elements c_{ij} in k and some paths q_{ij} in R . Since all the paths occurring in any x_i have the same length, we can assume without loss of generality that the paths q_{ij} all have the same length, say t . Since $J^t = \Pi_{q \in \mathcal{B}_t} q R$, fixing q in \mathcal{B}_t , it follows that $\sum_{q_{ij}=q} c_{ij} x_j = 0$, which implies that $\sum_{q_{ij}=q} c_{ij} x_j = 0$. By assumption, we have that $c_{ij} = 0$ for all $q_{ij} = q$. Hence we infer that $\sum_{i \in \mathcal{I}} R x_i$ is a direct sum. Similarly, $\sum_{i \in \mathcal{I}} x_i R$ is a direct sum. \square

We now show that the $\{f_i^n\}$ obtained from a right minimal projective resolution of Λ_0 and the $\{g_j^n\}$ obtained from a left minimal projective resolution of Λ_0 can be chosen to be the same.

Proposition 1.4. *Let $\Lambda = kQ/I$ be a Koszul algebra. Let $\{f_i^n\}_{i=0}^{t_n}$ and $\{g_i^n\}_{i=0}^{s_n}$ define a minimal resolution of Λ_0 as a right Λ -module and as a left Λ -module, respectively. Then $s_n = t_n$ for all $n \geq 0$ and the set $\{g_i^n\}_{i=0}^{t_n}$ can be chosen to be equal to the set $\{f_i^n\}_{i=0}^{t_n}$ for all $n \geq 0$.*

Proof. For n equal to 0, 1, or 2 the result is clear. Let $n \geq 3$. We proceed by induction on n and assume that the result is true for all $i < n$. By Theorem 1.1, for each i with $0 \leq i \leq t_n$ the equalities

$$f_i^n = \sum_{p,q} c_{pq} f_p^1 f_q^{n-1} = \sum_{p',q'} c_{p',q'} f_{p'}^2 f_{q'}^{n-2}$$

hold for some c_{pq} and $c_{p',q'}$ in k . Hence f_i^n is in $(\Pi_q R f_q^{n-1}) \cap (\Pi_{q'} I f_{q'}^{n-2})$, which, by induction, is equal to $(\Pi_{i=0}^{s_n} R g_i^n) \amalg (\Pi_i R g_i^{n'})$. Since $\sum_i f_i^n R$ is direct, the set

$\{f_i^n\}$ is linearly independent as vectors over k , and therefore the sum $\sum_i Rf_i^n$ is direct by Lemma 1.3. By length considerations, $\{f_i^n\}_{i=0}^{t_n}$ is contained in the k -linear span of $\{g_i^n\}_{i=0}^{s_n}$. Therefore $t_n \leq s_n$. By switching the roles of $\{f_i^n\}$ and $\{g_j^n\}$ and using the argument above, we conclude that $\{g_i^n\}_{i=0}^{s_n}$ is linearly independent and each g_i^n is in k -linear span of $\{f_i^n\}_{i=0}^{t_n}$. Hence $s_n = t_n$. By Lemma 1.3 it follows that $\Pi_{i=0}^{t_n} Rf_i^n = \Pi_{i=0}^{t_n} Rg_i^n$. This shows that we can choose the set $\{g_i^n\}_{i=0}^{t_n}$ equal to $\{f_i^n\}_{i=0}^{t_n}$. \square

Proposition 1.4 implies that, given a minimal projective resolution of Λ_0 as a right Λ -module in the form of $\{f_i^n\}$, we have all the information to construct a minimal projective resolution of Λ_0 as a left Λ -module. More precisely, take the $\{f_i^n\}$ as the $\{g_i^n\}$, and the maps in the left resolution are given by $g_i^n \mapsto \sum_{p=0}^{t_1} \sum_{q=0}^{t_{n-1}} c_{pq}(n, i, 1) g_p^1 g_q^{n-1}$.

2. A MINIMAL PROJECTIVE BIMODULE RESOLUTION OF Λ

In this section we turn our attention to the construction of a minimal projective Λ^e -resolution of Λ . This construction uses the comultiplicative structure of the minimal projective resolution of Λ_0 as a right Λ -module found in Theorem 1.1. This is applied to show an unpublished result of E. L. Green and D. Zacharia that Λ is a Koszul algebra if and only if Λ is a (right) linear module over Λ^e .

The following result also shows that the knowledge of the $\{f_i^n\}$ from a minimal projective resolution of Λ_0 as a right Λ -module is sufficient to explicitly give the projective modules and the differentials in a minimal projective resolution of Λ as a right Λ^e -module. The structure of the projective modules in a minimal projective resolution of Λ as right Λ^e -module was first given in [6]. Recall that the notation $\bar{*}$ denotes the natural residue class of $*$ modulo I . Let $\{c_{pq}(n, i, r)\}$ be as in Theorem 1.1.

Theorem 2.1. *Let $\Lambda = kQ/I$ be a Koszul algebra, and let $\{f_i^n\}_{i=0}^{t_n}$ define a minimal resolution of Λ_0 as a right Λ -module. A minimal projective resolution (\mathbb{P}, δ) of Λ over Λ^e is given by*

$$P^n = \Pi_{i=0}^{t_n} \Lambda \mathfrak{o}(f_i^n) \otimes_k \mathfrak{t}(f_i^n) \Lambda$$

for $n \geq 0$, where the j -th component of the differential $\delta^n: P^n \rightarrow P^{n-1}$ applied to the i -th generator $\mathfrak{o}(f_i^n) \otimes \mathfrak{t}(f_i^n)$ is given by

$$\sum_{p=0}^{t_1} c_{pj}(n, i, 1) \overline{f_p^1} \mathfrak{o}(f_j^{n-1}) \otimes \mathfrak{t}(f_j^{n-1}) + (-1)^n \sum_{q=0}^{t_1} c_{jq}(n, i, n-1) \mathfrak{o}(f_j^{n-1}) \otimes \mathfrak{t}(f_j^{n-1}) \overline{f_q^1}$$

for $j = 0, 1, \dots, t_{n-1}$ and $n \geq 1$, and $\delta^0: \Pi_{i=0}^{t_0} \Lambda e_i \otimes_k e_i \Lambda \rightarrow \Lambda$ is the multiplication map.

In particular, Λ is a linear module over Λ^e .

Proof. Direct computations show that $(\delta)^2 = 0$, so that (\mathbb{P}, δ) is a linear complex. In addition, note that $(\Lambda_0 \otimes_\Lambda \mathbb{P}, 1_{\Lambda_0} \otimes \delta)$ is a minimal resolution of Λ_0 as a right Λ -module.

In our setting, we have that $\Lambda^e / \text{rad } \Lambda^e \simeq \text{Hom}_k(\Lambda_0, \Lambda_0)$. Let (\mathbb{F}, d) be a minimal projective resolution of Λ as a right Λ^e -module. Then by [3, Chap. IX, Proposition

4.3] we have that

$$\begin{aligned} \operatorname{Hom}_{\Lambda^e/\operatorname{rad} \Lambda^e}(F^n/F^n \operatorname{rad} \Lambda^e, \Lambda^e/\operatorname{rad} \Lambda^e) &\simeq \operatorname{Ext}_{\Lambda^e}^n(\Lambda, \Lambda^e/\operatorname{rad} \Lambda^e) \\ &\simeq \operatorname{Ext}_{\Lambda}^n(\Lambda_0, \Lambda_0) \\ &\simeq \operatorname{Hom}_{\Lambda_0}(\Pi_{i=0}^{t_n} f_i^n R / \Pi_{i=0}^{t_n} f_i^n J, \Lambda_0) \end{aligned}$$

for all $n \geq 0$. In particular, $P^n \simeq F^n$ as Λ^e -modules for all $n \geq 0$, and hence $P^n/P^n \operatorname{rad} \Lambda^e \simeq F^n/F^n \operatorname{rad} \Lambda^e$ as $\Lambda^e/\operatorname{rad} \Lambda^e$ -modules for all $n \geq 0$. Note that these need not be isomorphic as graded modules, but, we in fact show that this is the case.

Since (\mathbb{P}, δ) is a complex, we obtain the following commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P^2 & \xrightarrow{\delta^2} & P^1 & \xrightarrow{\delta^1} & P^0 & \xrightarrow{\delta^0} & \Lambda & \longrightarrow & 0 \\ & & \downarrow \alpha^2 & & \downarrow \alpha^1 & & \downarrow \alpha^0 & & \parallel & & \\ \cdots & \longrightarrow & F^2 & \xrightarrow{d^2} & F^1 & \xrightarrow{d^1} & F^0 & \xrightarrow{d^0} & \Lambda & \longrightarrow & 0 \end{array}$$

Clearly $\alpha^0: P^0 \rightarrow F^0$ is an isomorphism, and we get an isomorphism $\alpha^0|_{\operatorname{Ker} \delta^0}: \operatorname{Ker} \delta^0 \rightarrow \operatorname{Ker} d^0$. Hence $\operatorname{Ker} \delta^0 / \operatorname{Ker} \delta^0 \operatorname{rad} \Lambda^e \simeq F^1/F^1 \operatorname{rad} \Lambda^e$. Since $\operatorname{Im} \delta^1$ is contained in $\operatorname{Ker} \delta^0$, this induces a map $\beta^1: P_1^1 \rightarrow \operatorname{Ker} \delta^0 / \operatorname{Ker} \delta^0 \operatorname{rad} \Lambda^e$. If β^1 is an isomorphism, then α^1 is an isomorphism and we have exactness at P^0 . Suppose that β^1 is not an isomorphism. Since P^1 is generated in degree 1, there is some projective summand of P^1 which is mapped to zero by δ^1 . Using the observation that $(\Lambda_0 \otimes_{\Lambda} \mathbb{P}, 1_{\Lambda_0} \otimes \delta)$ is a minimal resolution of Λ_0 as a right Λ -module, we obtain a contradiction. Hence β^1 is an isomorphism.

Since α^1 is an isomorphism, we can use the above argument replacing α^0 by α^1 to show that α^2 is an isomorphism and exactness at P^1 . By induction we infer that (\mathbb{P}, δ) is exact. Since the terms $\overline{f}_p^1 \mathfrak{o}(f_j^{n-1}) \otimes \mathfrak{t}(f_j^{n-1})$ and $\mathfrak{o}(f_j^{n-1}) \otimes \mathfrak{t}(f_j^{n-1}) \overline{f}_q^1$ are elements of degree one in Λ^e , we conclude that (\mathbb{P}, δ) is a minimal linear projective resolution of Λ over Λ^e . This also implies that Λ is a linear module over Λ^e . The proof is now complete. \square

As a consequence we obtain the next corollary which was first proved by E. L. Green and D. Zacharia.

Corollary 2.2. *Let $\Lambda = kQ/I$ be a graded algebra. Then Λ is a Koszul algebra if and only if Λ as a (right) Λ^e -module is a linear module.*

Proof. Suppose that Λ is a Koszul algebra. Then Theorem 2.1 implies that Λ has a linear projective Λ^e -resolution, and hence Λ is a linear module over Λ^e .

Suppose that Λ is a linear module over Λ^e as a right module. Let (\mathbb{P}, δ) be a linear projective resolution of Λ as a right Λ^e -module. Tensoring \mathbb{P} with Λ_0 , we obtain $\Lambda_0 \otimes_{\Lambda} \mathbb{P}$. But $\Lambda_0 \otimes_{\Lambda} \mathbb{P}$ is a linear projective resolution of Λ_0 as a right Λ -module. Hence Λ is a Koszul algebra, and we are done. \square

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EDWARD L. GREEN, DEPARTMENT OF MATHEMATICS, VIRGINIA TECH, BLACKSBURG, VA 24061, USA

E-mail address: `green@math.vt.edu`

GREGORY HARTMAN, DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ARIZONA, 617 N. SANTA RITA AVE., P.O. Box 210089, TUCSON, AZ 85721-0089, USA

E-mail address: `hartman@math.arizona.edu`

EDUARDO N. MARCOS, INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE SÃO PAULO (IME-USP), RUA DO MATÃO, 1010 - CIDADE UNIVERSITÁRIA, CEP 05508-090, SÃO PAULO - SP - BRAZIL

E-mail address: `enmarcos@ime.usp.br`

ØYVIND SOLBERG, INSTITUTT FOR MATEMATISKE FAG, NTNU, N-7491 TRONDHEIM, NORWAY

E-mail address: `oyvinso@math.ntnu.no`